Born-Infeld generalization of the Reissner-Nordström black hole

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Abstract

In this work we study the trajectories of test particles in a geometry that is the nonlinear electromagnetic generalization of the Reissner-Nordström solution. The studied spacetime is a Einstein-Born-Infeld solution, nonsingular outside a regular event horizon and characterized by three parameters: mass M, charge Q and the Born-Infeld parameter b related to the magnitude of the electric field at the origin. Asymptotically it is a Reissner-Nordström solution.

1 Introduction

The Reissner-Nordström (R-N) solution (characterized by its charge Q and mass M) turns out to be the final fate of a charged star, having as uncharged limit the Schwarzschild black hole, then it is of interest to investigate in more detail its nonlinear electromagnetic generalization.

Moreover, Einstein-Maxwell (EM) theory is a consistent truncation of the supergravity. Any asymptotically flat solution of EM theory with an extremal charge and vanishing angular momentum defines a BPS solution of supergravity [1].In particular, extremal ($Q = M$) Reissner-Nordström black hole is a BPS state in supersymmetry. States that are slightly out of BPS limit have interest in the study of black hole horizons, in questions like hairy black holes, isolated horizons [2] and entropy of black holes [3].

Besides, recently much attention has been deserved to nonlinear electrodynamics in string theory, where solutions of Born-Infeld equations represent states of D-branes [4].

In this paper we study the Einstein-Born-Infeld (EBI) solution that is the nonlinear electromagnetic generalization of the R-N black hole. One point of interest is the singularity that possesses this Einstein-Born-Infeld solution. Singularity that however, is not always reached by photons or test particles, this depending on the balancing between the three parameters mass, charge and the Born-Infeld parameter. Moreover, it turns out that the nonlinearity in the electromagnetic field modifies the size of the horizon as well as the effective geometry seen by the B-I photons. Basic facts on B-I nonlinear electrodynamics are sketched in Sec. 2. In Sec. 3 we present the EBI solution in appropriate coordinates and study the relevant metric function. In Sec. 4 the effective potentials are shown for test particles and photons. Some conclusions are drafted in the last section.

2 The Born-Infeld nonlinear electrodynamics

For completeness we include in this section the basic facts on nonlinear electrodynamics proposed by Born and Infeld [5] in the formalism implemented by Plebañski [6] for solutions of Petrov type D. The Born-Infeld electrodynamics is derived from an action for the gravitational electromagnetic field given by

$$S = \int d^4x\sqrt{-g}\{R(16\pi)^{-1} + L\}, \tag{1}$$

where R denotes the scalar curvature, $g := \det|g_{\mu\nu}|$ and L, the electromagnetic part, is assumed to depend on the so called structural function $K(P, \check{Q})$ constructed from the invariants of the electromagnetic field, P and $\check{Q}$,

$$L = -\frac{1}{2}P^{\mu\nu}F_{\mu\nu} + K(P, \check{Q}), \tag{2}$$

where $F_{\mu\nu}$ corresponds to the intensity of electric field and magnetic induction vectors (E and B) while the antisymmetric tensor $P^{\mu\nu}$ corresponds to the electric induction and the intensity of the magnetic field vectors (D and H). P and $\check{Q}$ are the two invariants of the tensor $P_{\mu\nu}$:

$$P := \frac{1}{4} P_{\mu\nu} P^{\mu\nu}, \check{Q} = \frac{1}{4} P_{\mu\nu} \check{P}^{\mu\nu}, \tag{3}$$

with $\check{P}_{\mu\nu} := -\frac{\epsilon_{\mu\nu\rho\sigma}}{2\sqrt{-g}} P^{\rho\sigma}$, $\epsilon_{\mu\nu\rho\sigma}$ is the totaly antisymmetric Levi-Civita tensor. The admissible structural functions $K(P, \check{Q})$ are constrained to fulfill the requirements of the correspondence to the linear theory ($K = P + O(P^2, \check{Q}^2)$), the parity conservation ($K(P, \check{Q}) = K(P, -\check{Q})$), the positive definitness of the energy density ($K_{,P} > 0$) and the requirement of the timelike nature of the energy flux vector ($PK_{,P} + \check{Q}K_{,\check{Q}} - K \geq 0$), where $K_{,P} := \frac{\partial K}{\partial P}$ and $K_{,\check{Q}} := \frac{\partial K}{\partial \check{Q}}$.

Performing the variation of S with respect to $g_{\mu\nu}$, A_μ and $P^{\mu\nu}$, the least action principle, $\delta S = 0$, lead to the dynamical equations, namely, Einstein equations, Maxwell-Faraday equations and the material or constitutive equations (that relate F_{ab} with P_{ab}).

Working in the null-tetrad formalism, the field structures to be studied are given by the metric

$$g = 2e^1 \otimes e^2 + 2e^3 \otimes e^4, \quad e^1 = \bar{e}^2, \quad e^3 = \bar{e}^3, \quad e^4 = \bar{e}^4, \tag{4}$$

Accompanied by the two-form of the nonlinear electromagnetic field,

$$\omega = \frac{1}{2}(F_{ab} + \check{P}_{ab}) e^a \wedge e^b, \tag{5}$$

where $\check{P}_{ab} := -\frac{1}{2}\epsilon_{abcd} P^{cd}$ and the tensor field F_{ab} is determined through the material equations

$$F_{ab} = K_{,P} P_{ab} + K_{,\check{Q}} \check{P}_{ab}. \tag{6}$$

The closure condition of the two-form ω is equivalent to the Maxwell and Faraday equations:

$$d\omega = 0 \to \check{F}^{ab}{}_{;b} = 0, \quad P^{ab}{}_{;b} = 0. \tag{7}$$

The system of equations for NLE is closed by the Einstein equations

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi E_{ab}, \tag{8}$$

where the energy-momentum tensor E_{ab} is given by

$$4\pi E_{ab} = K_{,P}(-P_{as}P_b{}^s + g_{ab}P) + (PK_{,P} + \check{Q}K_{,\check{Q}} - K)g_{ab}, \tag{9}$$

the scalar curvature R being

$$R = -8(PK_{,P} + \check{Q}K_{,\check{Q}} - K). \tag{10}$$

The Born-Infeld nonlinear electrodynamics is characterized by the structural function

$$K = b^2(1 - \sqrt{1 - \frac{2P}{b^2} + \frac{\check{Q}^2}{b^4}}), \tag{11}$$

the parameter b is the electric field at the origin ($r = 0$). The limiting transition $b \to \infty$ guarantees the correspondence to the linear Maxwell theory with $K = P$. The structural function in (11) is singled out among all possible structural functions by leading to a single family of characteristic surfaces [7].

For metrics of the Petrov type D one can always align the directions of the real null vectors, e^3, e^4, along the Debever-Penrose vectors. Also the eigenvectors of F_{ab} (and consequently the ones of P_{ab}) can be aligned in the directions of the Debever-Penrose vectors. These alignments leave as nonvanishing components of F_{ab} (P_{ab}), F_{12}, F_{34} (P_{12}, P_{34}).

3 Born-Infeld generalization of Reissner-Nordström solution

In this section some remarkable properties of the Born-Infeld (B-I) generalization of the Reissner-Nordström (R-N) solution are given.

This Einstein-Born-Infeld solution was presented in [8]. However, it was presented in terms of a not very friendly elliptic integral and in coordinates from which is not clear how to obtain its linear electromagnetic limit (R-N) neither its uncharged limit (Schwarzschild) (cf. Eqs. (4.11)-(4.16) in [8]). In the so-called canonical coordinates (t, r, θ, ϕ) in which the R-N metric is usually given [9] and writting the elliptic integral (which is of the first kind) in terms of the Legendre's elliptic functions $F(\beta, k) := \int_\beta^\infty (1 - k^2 \sin^2 s)^{-\frac{1}{2}} ds$, the solution is given by

$$
\begin{aligned}
ds^2 &= -\psi dt^2 + \psi^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), && (12)\\
\psi &= 1 - \frac{2M}{r} - \frac{\lambda r^2}{3} + \frac{2}{3} b^2 r^2 (1 - \sqrt{1 + \frac{Q^2}{b^2 r^4}}) + \\
& \frac{2Q^2}{3r}\sqrt{\frac{b}{Q}} F(\arccos\{\frac{br^2/Q - 1}{br^2/Q + 1}\}, \frac{1}{\sqrt{2}}), && (13)
\end{aligned}
$$

$$
F_{rt} = Q(r^4 + \frac{Q^2}{b^2})^{-\frac{1}{2}}, \quad (14)
$$

Where $G = c = 1$, M is the mass parameter, Q is the electric charge (both in lenght units) and b is the Born-Infeld parameter which corresponds to the magnitude of the electric field at $r = 0$, given in units of $[\text{lenght}]^{-1}$. The solution includes the cosmological constant λ, in such a manner that one of the limit cases is the de Sitter solution (when $M = b = Q = 0$). With the substitution $Q \to \sqrt{Q^2 + g^2}$, the solution includes the so-called magnetic charge g (remind that the Born-Infeld theory has the freedom of the duality rotations). In this work we put $\lambda = 0$ and $g = 0$.

As we state before, in nonlinear electromagnetism, the role of the skew symmetric field tensor $F_{\mu\nu}$ is now played by the tensor $P_{\mu\nu}$; both are related through the material or constitutive equations (6). For the Born-Infeld R-N solution $P_{\mu\nu}$ is given by

$$
P_{rt} = \frac{Q}{r^2}, \quad (15)
$$

The field P_{rt} is singular at the origin while F_{rt}, Eq. (14), at the origin gives a finite value corresponding to the magnitude of b. F_{rt} corresponds to the gradient of a potential A_t

$$
A_t(r) = Q \int_r^\infty (\frac{Q^2}{b^2} + s^4)^{-\frac{1}{2}} ds, \quad (16)
$$

The two nonvanishing components of the nonlinear electromagnetic Born-Infeld field, Eq. (9), in this case are given by

$$
8\pi E_{12} = 2b^2[(1 + \frac{Q^2}{b^2 r^4})^{-\frac{1}{2}} - 1],
$$

$$8\pi E_{34} = 2b^2[(1+\frac{Q^2}{b^2r^4})^{\frac{1}{2}}-1], \tag{17}$$

In the limit of large distances, $r \to \infty$, aymptotically it corresponds to the R-N solution. Also when the B-I parameter goes to infinity, $b \to \infty$, we recover the linear electromagnetic (Einstein-Maxwell) R-N solution, with

$$ds^2 = -\psi_{RN}dt^2 + \psi_{RN}^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{18}$$

$$\psi_{RN} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{19}$$

$$F_{rt} = \frac{Q}{r^2}, \tag{20}$$

To obtain these limits from Eqs. (13)-(14) we must consider the integral expression for the Legendre elliptic function F

$$F(\arccos\{\frac{br^2/Q-1}{br^2/Q+1}\}, \frac{1}{\sqrt{2}}) = 2\sqrt{\frac{Q}{b}}\int_r^\infty (\frac{Q^2}{b^2}+s^4)^{-\frac{1}{2}}ds. \tag{21}$$

In the limit $b \to \infty$ the fourth term of the metric function ψ, Eq. (13) is $-Q^2/3r^2$ while the last term amounts to $4Q^2/3r^2$; the sum of both terms amounts to Q^2/r^2, according with ψ_{RN}. Clearly, in the limit of $Q = 0$ the Schwarzschild solution is obtained. We observe that at $r = 0$ the last two terms in ψ_{RN} diverge. On the other side, for ψ we have that Eq. (13) can be written (with $\lambda = 0$) as

$$\begin{aligned}\psi &= 1 + \frac{2}{3}b^2r^2(1-\sqrt{1+\frac{Q^2}{b^2r^4}}) \\ &\quad + \frac{2}{r}\{-M + \frac{Q^2}{3}\sqrt{\frac{b}{Q}}F(\arccos\{\frac{br^2/Q-1}{br^2/Q+1}\}, \frac{1}{\sqrt{2}})\},\end{aligned} \tag{22}$$

the second term when evaluated at $r = 0$ is finite; the elliptic function F (cf. Eq. (21)) also is finite when evaluated at $r = 0$. Then, depending on the value of the term in curly brackets, we have three possible cases of behavior of ψ in the vicinity of $r = 0$:

(i) $M > \frac{Q^2}{3}\sqrt{\frac{b}{Q}}F(\arccos\{\frac{br^2/Q-1}{br^2/Q+1}\}, \frac{1}{\sqrt{2}})$, $\quad \psi$ diverges as $-\infty$ at $r \to 0$.

(ii) $M < \frac{Q^2}{3}\sqrt{\frac{b}{Q}}F(\arccos\{\frac{br^2/Q-1}{br^2/Q+1}\},\frac{1}{\sqrt{2}})$, ψ diverges as $+\infty$ at $r \to 0$.
(iii) $M = \frac{Q^2}{3}\sqrt{\frac{b}{Q}}F(\arccos\{\frac{br^2/Q-1}{br^2/Q+1}\},\frac{1}{\sqrt{2}})$, ψ does not diverge at $r \to 0$.

In spite that in case (iii) ψ is regular in all the range, the invariants diverge at $r = 0$, that is the conclusion from the analysis of the Weyl scalar Ψ_2. Since the solution is of type D, the only nonvanishing Weyl scalar is Ψ_2,

$$\Psi_2 = \frac{M}{r^3} - \frac{Q^2r^3}{6}\partial_r(\partial_r(\frac{1}{r^2}\int_r^\infty (s^2 + \sqrt{s^4 + \frac{Q^2}{b^2}})^{-1}ds)). \tag{23}$$

The invariants of this solution depend on Ψ_2^2, then at $r = 0$ there is a singularity at leats of the order $1/r^6$, comig from the first (mass) term, similar to the case of Schwarzschild and Reissner-Nordström. Furthermore, the second term in Eq. (23), due to the electromagnetic field, also diverges at $r = 0$. Then in this spacetime there is solely one singularity at $r = 0$.

The zeros of the metric function ψ indicate the existence of coordinate singularities, which can be eliminated by a change of coordinates. The entire analytic extension may consist of several of such patches, overlapping in finite regions. The procedure to follow to determine these analytic continuations is given by Graves and Brill in [10] (see also [9] for analytical extension of R-N metric). It consists in determining a simultaneous transformation of r and t to $[u(r,t), v(r,t)]$ in terms of which the light cones are lines with slopes ± 1, then in terms of these (u, v) coordinates the metric is regular. Such procedure can be used whenever the singularity in the metric function ψ is a zero of the first order. That turns out to be the case for the metric function ψ Eq. (13), since when the lower limit in the integral expression (21) is fixed, then the Legendre elliptic function is a constant.

The zeros of the metric function ψ have to be localized numerically. To this end we illustrate for several values of the parameters the behavior of the metric function ψ. In these plots ψ is in terms of the adimensional coordinate $u = r/M$, as

$$\begin{aligned}\psi &= 1 - \frac{2}{u} + \frac{2}{3}(bM)^2u^2(1 - \sqrt{1 + \frac{\alpha^2}{(bM)^2u^4}}) + \\ &\quad \frac{2\alpha^2}{3u}\sqrt{\frac{bM}{\alpha}}F(\arccos\{\frac{bMu^2/\alpha - 1}{bMu^2/\alpha + 1}\}, \frac{1}{\sqrt{2}}),\end{aligned} \tag{24}$$

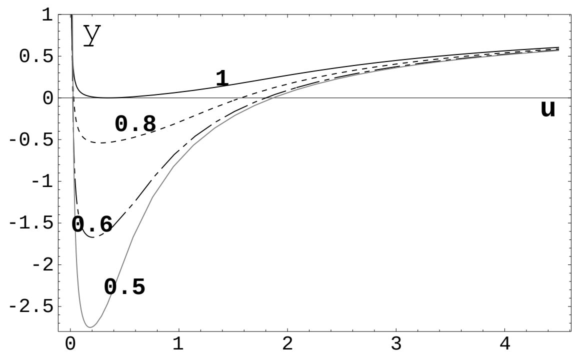


Fig. 1a. It is shown the shape of the metric function ψ vs. the coordinate $u = r/M$. The location of $r_h = u_h M$, with $\psi(r_h) = 0$, is the position of the horizon. The values of b were chosen such that at $u \to 0$, $\psi \to +\infty$. The value of $\alpha = Q/M$ is printed on each curve, the corresponding b's are: (α, b), $(0.5, 4.5/M)$, $(0.6, 2.5)$, $(0.8, 1.03)$, $(1, 0.5225)$.

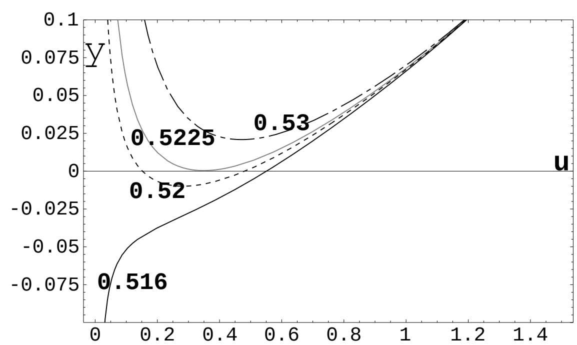


Fig. 1b. It is shown the behavior of the metric function ψ for the extreme case $(Q = M)$ with different values of b. For $b > 0.5225/M$ the function $\psi \geq 0$ and the metric is regular everywhere except at the origin. A naked singularity is present. If $b \leq 0.516/M$, the Schwarzschild behavior dominates.

In the plots we shall use, instead of Q, the constant α, $\alpha^2 = \frac{Q^2}{M^2}$, this constant measures the ratio between the charge and mass of the body that produces the field. b and M can vary independently and (bM) is adimensional. We identify the position of the horizon as the value for which g_{tt} $(= \psi)$ is zero. From the graphics for ψ (figs. 1a, 1b) we see that the zeros of this function can be one, two or not occur at all.

For $\alpha > 1$ (hyperextreme case, $Q > M$) $\psi = 0$ has no zeros, as in the R-N case . For small α's ($\alpha < 0.4$) the behavior resembles the Schwarzschild one, independently of the value of b. In this case the gravitational field due to the mass M overwhelms the (linear or nonlinear) electromagnetic field.

For values of $0.5 < \alpha < 0.9$, the metric function resembles Schwarzschild if $b < 0.7$ (weak electromagnetic field), see Fig. 1a. For these values, $0.5 < \alpha < 0.9$, the magnitude of bM can be adjusted in order that ψ grews to $+\inf ty$ near $r = 0$. In these cases ψ has two zeros. In Fig. 1a it is also apparent that as greater is α, smaller is the value of b that is sufficient to defeat the gravitational attraction at $r = 0$. The reason is that as α grows, then the mass diminish in relation to the charge, and a smaller nonlinear electromagnetic field (smaller b) overwhelms the gravitational one.

For α, $\alpha > 0.5$ and $b > 4.5/M$, ψ goes to infinity ($+\infty$) at $r \to 0$, resembling a soliton like behavior. Furthermore, as greater is b, the point where ψ becomes zero, is nearer the origin, i.e. the size of the horizon stretches (see fig. 1b).

Special attention deserves the case $\alpha = 1$, that corresponds in the black hole terminology to the extreme case ($Q = M$). In fig. 1b., for $\alpha = 1$, the function ψ shows a very sensitive behavior with respect to the value of b in the neighborhood of $b = 0.5/M$. In the vicinity of this value three cases can occur: one horizon, two horizons or not horizon at all. When $\alpha = 1$ and $b = 0.5225/M$ the metric function ψ possesses one horizon and goes to $+\infty$ near $r = 0$. If $\alpha = 1$ and $b \geq 0.53/M$ then ψ has no zeros and presents a naked singularity. For $\alpha = 1, b < 0.516M$, gravity dominates and the behavior is Schwarzschild-like.

We pass now to describe some features of the trajectories of test particles.

4 Trajectories of test particles, photons and gravitons

As in the Schwarzschild and R-N case, the timelike and null geodesics in the equatorial plane ($\theta = \frac{\pi}{2}$) of the B-I spacetime reduces to solving a problem of ordinary one-dimensional motion in an effective potential.

4.1 Charged and uncharged test particles

Here we examine the law of motion of test particles of mass μ and charge ϵ. The geodesic equation is

$$\frac{d^2x^\nu}{d\tau^2} + \Gamma_{\alpha\beta}{}^{\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = -\frac{\epsilon}{\mu}F_\sigma{}^{\nu}\frac{dx^\sigma}{d\tau}, \tag{25}$$

The test particle has two conserved quantities at least, corresponding to the two Killing vectors ∂_t and ∂_ϕ, i.e. the energy of the particle and its angular momentum, $\tilde{l}$. If the test particle is charged, then the energy involves an electromagnetic part that arises from the right hand side of Eq. (25), corresponding to the only nonvanishing component of the electromagnetic field, F_{rt}. The geodesic equation for the t coordinate amounts to

$$\ddot{t} + \psi^{-1}(\psi_{,r})\dot{r}\dot{t} = \frac{\epsilon}{\mu}\frac{F_{rt}}{\psi}\dot{r}, \tag{26}$$

where dot indicates the derivative with respect to an affine parameter. This equation can be integrated for $\dot{t}$, obtaining,

$$\begin{aligned} \dot{t}\psi &= E + \frac{\epsilon}{\mu}\int_r^\infty F_{rt}dr, \\ &= E + \frac{\epsilon Q}{\mu}\int_r^\infty (r'^4 + \frac{Q^2}{b^2})^{-\frac{1}{2}}dr' \\ &= E + \frac{\epsilon Q}{\mu}\sqrt{\frac{b}{Q}}\frac{1}{2}F(\arccos\{\frac{r^2 - Q/b}{r^2 + Q/b}\}, \frac{1}{\sqrt{2}}). \end{aligned} \tag{27}$$

where E is the energy at infinity of the uncharged test particle. On the other hand, from the line element for the timelike geodesics we have,

$$1 = \psi\dot{t}^2 - \psi^{-1}\dot{r}^2 - r^2\dot{\phi}^2, \tag{28}$$

Substituting $\tilde{l} = g_{\phi\phi}\dot{\phi}$, and $\dot{t}$ from Eq. (27), we obtain

$$\dot{r}^2 + \psi(\frac{\tilde{l}^2}{r^2} + 1) - [E + \frac{\epsilon Q}{2\mu}\sqrt{\frac{b}{Q}}F(\arccos\{\frac{r^2 - Q/b}{r^2 + Q/b}\}, \frac{1}{\sqrt{2}})]^2 = 0, \tag{29}$$

Comparing with $\frac{1}{2}\dot{r}^2 + V_{eff}(E, \tilde{l}, r) = 0$, we get the effective potential, which depends nontrivially on E as well as on $\tilde{l}$, for a charged test particle. The plots of V_{eff} are in terms of the adimensional coordinate u, and $l = \tilde{l}/M$,

$$V_{eff}(u) = \frac{\psi}{2}(\frac{l^2}{u^2} + 1) - [E + \frac{\epsilon\alpha}{2\mu}\sqrt{\frac{bM}{\alpha}}F(\arccos\{\frac{u^2 - \frac{\alpha}{bM}}{u^2 + \frac{\alpha}{bM}}\}, \frac{1}{\sqrt{2}})]^2/2, \quad (30)$$

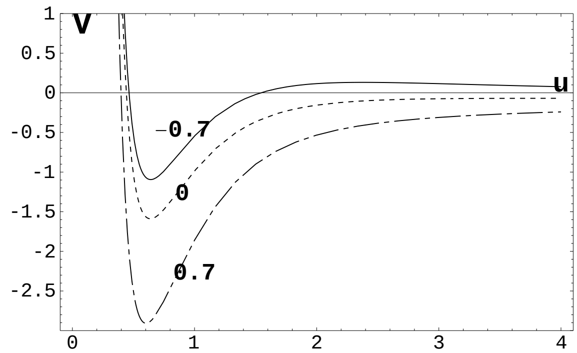


Fig. 2a. The effective potential, V, for test particles with different ratio charge/mass, ϵ/μ, as printed over each plot, $\epsilon/\mu = -0.7, 0, 0.7$. The values of the constants are $E = 1, l = 3, \alpha = 0.9, b = 0.9/M$. The change in sign of the charge results in a quantitative difference. There are stable equilibrium positions that do not depend much on the ratio ϵ/μ.

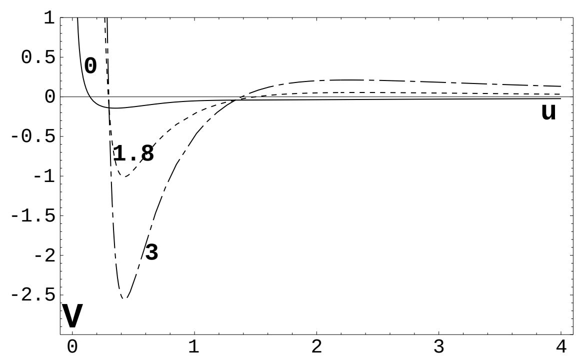


Fig. 2b. The effective potential, V, for test particles varying the angular momentum, ($l = \tilde{l}/M$, where $\tilde{l}$ is the angular momentum of the test particle), as printed, $l = 0, 1.8, 3$. The values of the constants are $E = 1, \epsilon/\mu = -1, \alpha = 0.9, b = 0.8/M$. There are stable equilibrium positions of lower energy for greater l.

In Fig. 2a it is shown the effective potential for different charge/mass, $\frac{\epsilon}{\mu}$, values of the test particle. The shape of the potential is the same independently if the particle has positive, negative or not charge at all. There is an a ttractive region with stable equilibrium positions. Fig. 2b corresponds to the variation of the angular momentum of the test particle.

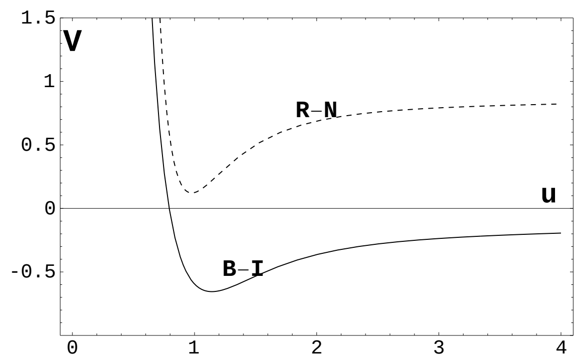


Fig. 3 In this figure the effective potential as felt by a test particle in R-N and in B-I is compared in the extreme case ($M = Q$). The constants are $\epsilon/\mu = 0.5, E = 1, l = 3$ and for B-I, $b = 0.8/M$.

In Fig. 3 the comparison between the Born-Infeld and Reissner-Nordström effective potentials is shown in the extreme case ($Q = M$). We remind that in R-N spacetime, test particles do not reach the singularity at $r = 0$, while photons always do. In B-I spacetime, both behaviors are possible: to reach or not the singularity, depending on the value of the B-I parameter b. In particular, test particles can not scape the singularity if the metric function ψ goes to $-\infty$ when $r \to 0$.

4.2 Photons and Gravitons

In nonlinear electromagnetism photons do not propagate along null geodesics of the background geometry. Instead, photons propagate along null geodesics of an effective geometry which depends on the nonlinear theory (see [6]). Recently Novello it et al have presented studies on these lines [11] and also Gibbons has addressed this topic in the context of string theory [12].

The discontinuities of the field propagate obeying the equation for the characteristic surfaces, what in ordinary optics are the so called "eikonal

equations". For a curved spacetime the equation for the characteristic surfaces is

$$g^{\mu\nu} S_{,\mu} S_{,\nu} = 0. \tag{31}$$

The corresponding trajectories of the "rays" are the null geodesics. When nonlinear electrodynamics is involved, the corresponding equation is (cf. Eq. (10.106) in [6])

$$(g^{\mu\nu} + \frac{4\pi}{b^2} E^{\mu\nu}) S_{,\mu} S_{,\nu} = \gamma^{\mu\nu} S_{,\mu} S_{,\nu} = 0, \tag{32}$$

Eq.(32) shows that it is the energy momentum density, $E_{\mu\nu}$ (Eq. (9)), of the nonlinear field which is the essential cause why these surfaces are in general not null surfaces obeying Eq. (31). Then when one conside rs the effective potential corresponding to photons, instead of using the elements of $g_{\mu\nu}$, we must use the ones of the effective metric $\gamma_{\mu\nu}$. Eq. (31) then governs the propagation of the gravitational discontinuities, while Eq.(32) governs the propagation of nonlinear electromagnetic discontinuities. Those surfaces are locally normal surfaces to the trajectories of gravitons and photons, respectively. The linear (Maxwell) case is obtained when $b \to \infty$: the second term on the left hand side of Eq. (32) vanishes and Eq. (31) is recovered.

The trajectories corresponding to null rays are obtained from the line element

$$0 = \psi \dot{t}^2 - \psi^{-1} \dot{r}^2 - r^2 \dot{\phi}^2, \tag{33}$$

Substituting $E = -g_{tt}\dot{t} = \psi \dot{t}$ and $\dot{\phi} = \tilde{l}/r^2$ and comparing with $\dot{r} + V_{\rm g} = 0$ we have

$$V_{\rm g} = -\frac{g_{tt} l^2}{2u^2} = \frac{\psi l^2}{2u^2}, \tag{34}$$

while for the photons the corresponding effective potential must be obtained, in analogous way, from the effective metric $\gamma_{\mu\nu}$,

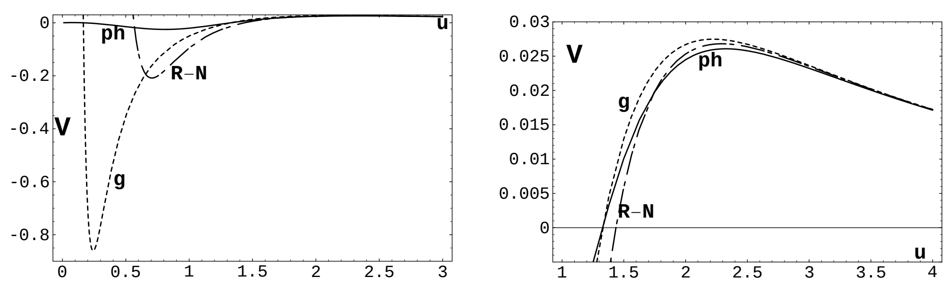
$$V_{\rm ph} = \frac{\psi l^2}{2u^2} (1 + \frac{\alpha^2}{(bM)^2 u^4})^{-1}, \tag{35}$$

The effective potential for photons is the one for gravitons but modulated by the factor $(1+\frac{\alpha^2}{(bM)^2u^4})^{-1}=(1+\frac{Q^2}{b^2r^4})^{-1}$. This factor goes to one in the linear case ($b\to\infty$) and we recover Eq. (34).

Eq. (35)can be written as

$$
\begin{aligned}
V_{\mathrm{ph}}(r) &= \frac{\tilde{l}^2}{2r^2}(1+\frac{Q^2}{b^2r^4})^{-1}, \\
&= \frac{\tilde{l}^2}{2}\{r^2-2Mr+\frac{2b^2}{3}[r^4-\sqrt{r^8+\frac{Q^2r^4}{b^2}}] \\
&\quad +\frac{2}{3}\sqrt{\frac{b}{Q}}Q^2rF(\arccos\{\frac{br^2/Q-1}{br^2/Q+1}\},\frac{1}{\sqrt{2}})\}, \qquad (36)
\end{aligned}
$$

from the last expression it is evident that at $r=0$ the effective potential for the photon is zero, $V_{\mathrm{ph}}(0)=0$. In contrast, when approaching the origin, the graviton potential V_{g} grows to plus infinity ($+\infty$), resembling a repulsive particle-like potential (in analogy with R-N). However, for the regions where the radial coordinate is $r<r_h$, with $\psi(r_h)=0$, we must keep in mind that r is actually a timelike coordinate, hence, it could be misleading to think on region $r<r_h$ as the *interior* of the black hole (excepting cases of double root when $\alpha=1$).



Figs. 4 It is shown the effective potential corresponding to null geodesics in R-N and B-I. In the latter case it is different the effective potential felt by a photon (ph) and the one felt by a graviton (g, null geodesics). The points where $V(u)=0$ corresponds to the horizon and it is shared by photon and graviton in the B-I case; it is closer to the origin than in the R-N case (the B-I field srinks the horizon). The values of the constants and parameters are: $\alpha=0.9, b=0.75/M, l=1$. The plots correspond to different ranges of u.

The different behavior is compared in Fig. 4 for both effective potentials and also compared with the Reissner-Nordström null trajectories for the same value of the charge-mass relation, $Q/M = .9$, and the same angular momentum ($l = 1$). For this plot the Born-Infeld parameter is $b = .75/M$. Of course gravitons interact with the B-I field but just through the effect of B-I field on the spacetime geometry.

5 Conclusions

We have studied the nonlinear electromagnetic generalization (Born-Infeld) of the Reissner-Nordström metric. It describes a nonsingular, asymptotically flat spacetime outside a regular event horizon. The solution is interesting because a straight comparison with R-N can be established. The effect of increasing the Born-Infeld parameter b is to srink the size of the horizon. Solutions without or with one or two horizons are possible, depending on the values of the parameters. Another feature of this solution is that, while for R-N all photons reach the singularity at $r = 0$, for the Born-Infeld generalization, some of them skip the singularity at $r = 0$.

Asymptotically this solution is a Reissner-Nordström one and then the global charges defined at spatial infinity such as ADM mass and electric charge are the global parameters that describe this solution. However the mass and electric charge do not determine completely the solution. For each solution characterized by the parameters (M, Q), there exist an infinite number of solutions with different Born-Infeld parameter b with a different behavior near the horizon (see Fig. 1b, for instance). In this sense this is a colored black hole, with an Abelian (Born-Infeld) hair. Further investigation is needed to determine if the solution is unstable.

In spite that the geometry for photons and gravitons is not the same, both share the same horizon (see Fig. 4, same $\psi(u) = 0$). This feature confirms for black holes, the conjecture claimed by Gibbons [12] in the string context: *if the closed string metric is static and the Born-Infeld field is pure electric or pure magnetic, then the open string metric can not have a non-singular event horizon distinct from the one given by the closed string metric*; it considering that gravitons obey the closed string action while photons the open string one, and that the solution studied here has a pure electric B-I field.

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